KASDI MERBAH UNIVERSITY-OUARGLA 2024-2025

1ST YEAR MASTER : FUNCT. ANAL., MOD. & NUM. ANAL. THEORY OF DISTRIBUTIONS

EXAM DURATION 01^h30^{mn}

EXERCISE 1. (04 PTS)

Let Ω is a bounded open set of $\mathbb{R}^N$ and $p \in [1, \infty[$.
– Show that the following condition :

$$f_n \to f \text{ in } L^p(\Omega)$$

is sufficient for $f_n \to f$ in $D'(\Omega)$.

EXERCISE 2. (08 PTS)

For all $\varphi \in \mathscr{D}(\mathbb{R})$, we set

$$\varphi \mapsto < pf(x^{-2}), \varphi >= \lim_{\varepsilon \to 0} \left[\int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x^2} dx - 2\frac{\varphi(0)}{\varepsilon} \right].$$

1. Show that $pf(x^{-2})$ is a distribution. What is its order ?
2. Define the distribution $pf(x^{-3})$. What is its order ?
3. Calculate $xpf(x^{-3})$.

EXERCISE 3. (08 PTS)

1. Let $T \in S'(\mathbb{R}^N)$ be a tempered distribution. Show that

$$\widehat{x^\alpha T} = i^{|\alpha|} \partial^\alpha \widehat{T}.$$

2. A distribution $u \in \mathscr{D}'(\mathbb{R}^N)$ is said to be homogeneous of degree $p \in \mathbb{R}$ if, for all $\lambda > 0$ and all $\varphi \in \mathscr{D}(\mathbb{R}^N)$, we have

$$< u, \varphi_\lambda >= \lambda^{-N-p} < u, \varphi >,$$

where : $\varphi_\lambda \in \mathscr{D}(\mathbb{R}^N)$ is defined by

$$\varphi_\lambda(x) = \varphi(\lambda x).$$

2.1. Show that on $\mathbb{R}^N$, the Dirac mass at 0 is homogeneous of degree $(-N)$.

2.2. Let $T \in S'(\mathbb{R}^N)$ be an homogeneous tempered distribution of degree p. Show that its Fourier transform $\widehat{T}$ is homogeneous. What is its degree ?

Exam: ThD 2024–2025

Cor. 1

We suppose that $f_n \to f$ in $L^p(\Omega)$. 0,5 Let $\varphi \in \mathcal{D}(\Omega)$, we have

$$
\begin{aligned}
|\langle f_n, \varphi\rangle - \langle f, \varphi\rangle| &= \left|\int_\Omega f_n(x)\varphi(x)\,dx - \int_\Omega f(x)\varphi(x)\,dx\right| \\
&= \left|\int_\Omega (f_n(x) - f(x))\,\varphi(x)\,dx\right| \\
&\leqslant \int_\Omega |(f_n(x) - f(x))\,\varphi(x)|\,dx. \quad 1
\end{aligned}
$$

According to Hölder's inequality, 0,5 we obtain

$$|\langle f_n, \varphi\rangle - \langle f, \varphi\rangle| \leqslant \|\varphi\|_{L^{p'}(\Omega)} \, \|f_n - f\|_{L^p(\Omega)} \xrightarrow[n\to 0]{} 0 \quad 1,5$$

then $f_n \to f$ in $\mathcal{D}'(\Omega)$. 0,5

Cor. 2

1. Let $\varphi(x) = \varphi(0) + x\varphi'(0) + x^2\psi(x)$, where $\|\psi\|_\infty \leqslant C\|\varphi''\|_\infty$ and $Supp(\varphi) = [-M, M]$.

We have

$$\langle pf(x^{-2}), \varphi\rangle = \lim_{\varepsilon\to 0}\left[\varphi(0)\int_{\varepsilon\leqslant|x|\leqslant M}\frac{1}{x^2}dx + \varphi'(0)\int_{\varepsilon\leqslant|x|\leqslant M}\frac{1}{x}dx + \int_{\varepsilon\leqslant|x|\leqslant M}\psi(x)dx - 2\frac{\varphi(0)}{\varepsilon}\right].$$

Since $\displaystyle\varphi(0)\int_{\varepsilon\leqslant|x|\leqslant M}\frac{1}{x^2}dx = 2\varphi(0)\int_\varepsilon^M\frac{1}{x^2}dx = \frac{2\varphi(0)}{\varepsilon} - \frac{2\varphi(0)}{M}$, 1

and $\displaystyle\varphi'(0)\int_{\varepsilon\leqslant|x|\leqslant M}\frac{1}{x}dx = 0$ ($x \mapsto \frac{1}{x}$ is odd).

Then
$$
\begin{aligned}
\langle pf(x^{-2}), \varphi\rangle &= \lim_{\varepsilon\to 0}\left[-\frac{2\varphi(0)}{M} + \int_{\varepsilon\leqslant|x|\leqslant M}\psi(x)dx\right] \quad 1 \\
&= \int_{|x|\leqslant M}\psi(x)dx - \frac{2\varphi(0)}{M}.
\end{aligned}
$$

Therefore, we deduce that $\displaystyle|\langle pf(x^{-2}), \varphi\rangle| \leqslant C\sum_{j=0}^{2}\|\varphi^{(j)}\|_\infty$ 1

Hence $pf(x^{-2})$ is a distribution of order $\leqslant 2$.

2. Let $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\operatorname{Supp}(\varphi) \subset [-M, M]$ and

$$\varphi(x) = \varphi(0) + x\varphi'(0) + \frac{x^2}{2}\varphi''(0) + x^3\psi(x), \text{ where } \|\psi\|_\infty \leq C\|\varphi^{(3)}\|_\infty$$

We set $$\langle pf(x^{-3}), \varphi\rangle = \lim_{\varepsilon \to 0} \Big[\int_{|x|\geq\varepsilon} \frac{\varphi(x)}{x^3}dx + R(\varepsilon)\Big].$$

We have

$$\int_{\varepsilon\leq|x|\leq M} \frac{\varphi(x)}{x^3}dx = \varphi(0)\int_{\varepsilon\leq|x|\leq M}\frac{1}{x^3}dx + \varphi'(0)\int_{\varepsilon\leq|x|\leq M}\frac{1}{x^2}dx + \frac{1}{2}\varphi''(0)\int_{\varepsilon\leq|x|\leq M}\frac{1}{x}dx + \int_{\varepsilon\leq|x|\leq M}\psi(x)dx.$$

Since $\displaystyle\int_{\varepsilon\leq|x|\leq M}\frac{1}{x^3}dx = \int_{\varepsilon\leq|x|\leq M}\frac{1}{x}dx = 0$ ($x \mapsto \frac{1}{x^3}$ and $x \mapsto \frac{1}{x}$ are odd),

and $$\varphi'(0)\int_{\varepsilon\leq|x|\leq M}\frac{1}{x^2}dx = 2\varphi'(0)\int_\varepsilon^M \frac{1}{x^2}dx = -2\frac{\varphi'(0)}{M} + 2\frac{\varphi'(0)}{\varepsilon},$$ 1

Then $R(\varepsilon) = -2\dfrac{\varphi'(0)}{\varepsilon}$.

We deduce that

$$\langle pf(x^{-3}), \varphi\rangle = \lim_{\varepsilon\to 0}\Big[\int_{|x|\geq\varepsilon}\frac{\varphi(x)}{x^3}dx - 2\frac{\varphi'(0)}{\varepsilon}\Big].$$ 1

Therefore $$|\langle pf(x^{-3}), \varphi\rangle| = \Big|-\frac{2\varphi'(0)}{M} + \int_{|x|\leq M}\psi(x)dx\Big| \leq C\sum_{j\leq 3}\|\varphi^{(j)}\|_\infty.$$ 1

Hence $pf(x^{-3})$ is a distribution of order ≤ 3.

3. We have

$$\langle pf(x^{-3}), \varphi\rangle = \lim_{\varepsilon\to 0}\Big[\int_{|x|\geq\varepsilon}\frac{\varphi(x)}{x^3}dx - \frac{2\varphi'(0)}{\varepsilon}\Big].$$

Therefore

$$\langle x\, pf(x^{-3}), \varphi\rangle = \langle pf(x^{-3}), x\varphi\rangle$$
$$= \lim_{\varepsilon\to 0}\Big[\int_{|x|\geq\varepsilon}\frac{\varphi(x)}{x^2}dx - 2\frac{(x\varphi)'(0)}{\varepsilon}\Big].$$ 0.5

Since $(x\varphi)'(0) = (\varphi + x\varphi')(0) = \varphi(0)$, we obtain 0.5

$$\langle x\, pf(x^{-3}), \varphi\rangle = \lim_{\varepsilon\to 0}\Big[\int_{|x|\geq\varepsilon}\frac{\varphi(x)}{x^2}dx - 2\frac{\varphi(0)}{\varepsilon}\Big] = \langle pf(x^{-2}), \varphi\rangle$$ 1

Then $x\, pf(x^{-3}) = pf(x^{-2})$.

Cor. 3

1. Let $\varphi \in S(\mathbb{R}^N)$, we have

$$\langle \widehat{x^\alpha T}, \varphi \rangle = \langle x^\alpha T, \hat{\varphi} \rangle = \langle T, x^\alpha \hat{\varphi} \rangle = \langle T, x^\alpha \int_{\mathbb{R}^N} e^{-iy.x} \varphi(y)\,dy \rangle$$

$$= \langle T, i^{|\alpha|} \int_{\mathbb{R}^N} (-i)^{|\alpha|} x^\alpha e^{-iy.x} \varphi(y)\,dy \rangle$$

$$= i^{|\alpha|} \langle T, \int_{\mathbb{R}^N} (\partial^\alpha e^{-iy.x}) \varphi(y)\,dy \rangle \quad 1$$

$$= i^{|\alpha|} \langle T, (-1)^{|\alpha|} \int_{\mathbb{R}^N} e^{-iy.x} (\partial^\alpha \varphi(y))\,dy \rangle \quad 0,5$$

$$= i^{|\alpha|} \langle T, (-1)^{|\alpha|} \widehat{\partial^\alpha \varphi} \rangle \quad 0,5$$

$$= i^{|\alpha|} (-1)^{|\alpha|} \langle \hat{T}, \partial^\alpha \varphi \rangle \quad 0,5$$

$$= i^{|\alpha|} (-1)^{|\alpha|} (-1)^{|\alpha|} \langle \partial^\alpha \hat{T}, \varphi \rangle \quad 0,5$$

$$= i^{|\alpha|} \langle \partial^\alpha \hat{T}, \varphi \rangle .$$

2.1. We have

$$\langle \delta_0, \varphi_\lambda \rangle = \varphi_\lambda(0) = \varphi(\lambda . 0) = \varphi(0) = \langle \delta_0, \varphi \rangle = \lambda^{-N-(-N)} \langle \delta_0, \varphi \rangle .$$

then δ_0 is homogenous of degree $(-N)$. 1

2.2 If $T \in S'(\mathbb{R}^N)$ is an homogeneous tempered distribution of degree p, then for all $\varphi \in S(\mathbb{R}^N)$ we obtain

$$\langle T, \varphi_\lambda \rangle = \lambda^{-N-p} \langle T, \varphi \rangle .$$

0,5

We have $\langle \hat{T}, \varphi_\lambda \rangle = \langle T, \widehat{\varphi_\lambda} \rangle$.

$$\widehat{\varphi_\lambda}(\xi) = \int_{\mathbb{R}^N} e^{-ix.\xi} \varphi(\lambda x)\,dx = \lambda^{-N} \int_{\mathbb{R}^N} e^{-iy.\frac{\xi}{\lambda}} \varphi(y)\,dy$$

$$= \lambda^{-N} \hat{\varphi}\left(\frac{\xi}{\lambda}\right) = \lambda^{-N} \hat{\varphi}_{\frac{1}{\lambda}} . \quad 1$$

therefore

$$\langle \hat{T}, \varphi_\lambda \rangle = \langle T, \lambda^{-N} \hat{\varphi}_{\frac{1}{\lambda}} \rangle = \lambda^{-N} \langle T, \hat{\varphi}_{\frac{1}{\lambda}} \rangle$$

$$= \lambda^{-N} \left(\frac{1}{\lambda}\right)^{-N-p} \langle T, \hat{\varphi} \rangle$$

$$= \lambda^{-N-(-N-p)} \langle \hat{T}, \varphi \rangle . \quad 2$$

then $\hat{T}$ is homogeneous of degree $(-N-p)$. 0,5