Module: CAF

TD 3. Strong Convergence - Weak Convergence - Weak star Convergence

Exercise 1. *Show that, if the dual of a Banach space X is reflexive, then X itself is reflexive.*

Exercise 2. *Let Y be a closed subspace of a reflexive Banach space X. Show that the quotient space X/Y is reflexive.*

Exercise 3. *(1) Let X be a reflexive Banach space. Show that, given any $x' \in X'$, there exists $x_0 \in X$ such that $\|x_0\| = 1$ and $\|x'\| = \sup_{\|x\|=1} |x'(x)| = x'(x_0)$.*

(2) Show that, conversely, if a Banach space X is such that, given any $x' \in X'$, there exists $x_0 \in X$ such that $\|x_0\| = 1$ and $\|x'\| = \sup_{\|x\|=1} |x'(x)| = x'(x_0)$, then X is réflexive.

Exercise 4. *Let $(X, \|.\|)$ be a reflexive Banach space, and let Z be a nonempty closed convex subset of X.*

1. *Show that, given any element $x \in X$, there exists $y \in Z$ such that $\|x - Y\| = \inf_{z \in Z} \|x - z\|$. Hint: Consider an infimizing sequence and use the Banach-Eberlein-Smulian theorem.*
2. *Show that y is unique if $(X, \|.\|)$ is strictly convex.*

Exercise 5. *Let the subset Z of the space $\mathcal{C}([0,1])$ equipped with the sup norm $\| \|$ be defined by*

$$Z = \left\{ f \in \mathcal{C}([0,1]); \int_0^{1/2} f(x)dx = 1 + \int_{1/2}^1 f(x)dx \right\}.$$

1. *Show that Z is a nonempty closed convex subset of $\mathcal{C}([0,1])$.*
2. *Show that $\inf_{f \in Z} \|f\| = 1$ but that there is no $f \in Z$ such that $\|f\| = 1$.*
3. *Conclude that the Banach space $(\mathcal{C}([0,1]), \|.\|)$ is not reflexive.*

Exercise 6. *Let X be a separable Banach space. Show that any bounded sequence in X' contains a weakly-$*$ convergent subsequence.*

Exercise 7. *Let Ω be an open subset of $\mathbb{R}^n$ and let $(f_k)_{k\geq 1}$ be a bounded sequence in $L^\infty(\Omega)$. Show that there exist a subsequence $(f_\sigma(k))_{k\geq 1}$ and a fonction $f \in L^\infty(\Omega)$ such that for each $g \in L^1(\Omega)$,*

$$\int_\Omega f_{\sigma(k)} g dx \to \int_\Omega f g dx \text{ as } k \to \infty.$$

Exercise 8. *Let X be a normed vector space. Show that the image $J(X)$ of X under the canonical isometry J is closed in X'' if and only if X is a Banach space.*

Exercise 9. *Let Ω be an open subset of $\mathbb{R}^n$, $1 < p < \infty$ and let functions $f_k \in L^p(\Omega)$, $k \geq 1$ and $f \in L^p(\Omega)$ be such that the sequence $(f_k)_{k\geq 1}$ is bounded in $L^p(\Omega)$ and f_k converges almost everywhere in Ω to f as $k \to \infty$. Show that*

$$f_k \rightharpoonup f \text{ as } k \to \infty.$$