Module: CAF

TD 2 Fundamentals Theorems of Functional Analysis

Exercise 1. *Given X a real Banach space, Y a proper closed subspace of X, and $x_0 \in X$ with $x_0 \notin Y$. Show that there exists $f \in X'$ such that $\|f\| = 1$, $f(x_0) = \text{dist}(x_0, Y)$, and $f_{|Y} = 0$.*

Exercise 2. ***Characterization of Adherent Points to a Subspace*** *Let X be a normed vector space, Y a subspace of X, and $z \in X$. Prove that z is adherent to Y if and only if any continuous linear functional on X that is zero on Y is also zero at z.*

Consequences:

1. *If $x \in X$ such that, for all $f \in X'$, $f(x) = 0$, then $x = 0$.*
2. *If every continuous linear functional that is zero on Y is also zero on X, then Y is dense in X.*

Exercise 3. *Consider the subspace $Y = \{x \in \mathbb{R}^3 : x_3 = 0\}$ of $\mathbb{R}^3$ and a linear functional $f : Y \to \mathbb{R}$ defined as $f(x) = a \cdot x$, where $a = (a_1, a_2, 0) \in \mathbb{R}^3$. Determine all linear extensions of f to $\mathbb{R}^3$.*

Exercise 4. *Let Y be the subspace of $C([0;1])$ consisting of constant functions. Provide an example of a linear and bounded functional on Y that has infinitely many linear extensions on $C([0;1])$ with the same norm as that of f.*

Exercise 5. *Let f be a bounded linear functional defined on a non-empty subspace $Y \neq \{0\}$ of a Hilbert space H. Show that f has a linear extension $\tilde{f}$ on H such that $\|f\| = \|\tilde{f}\|$.*

Exercise 6. *For a given semi-norm ρ on a vector space X, prove that for every $x_0 \in X$, there exists a functional f on X such that $f(x_0) = \rho(x_0)$ and $|f(x)| \leq \rho(x)$ for all $x \in X$.*

Exercise 7. *Suppose X is a normed vector space. Show that if $f(x) = f(y)$ for all $f \in X'$, then $x = y$.*

Exercise 8. *Prove that every subspace of a reflexive Banach space is also reflexive.*

Exercise 9. *Show that a Banach space is reflexive if and only if its dual is reflexive.*

Exercise 10. *Let X be a normed vector space, and M a subset of X. Prove that for any $x \in X$, x belongs to the closure of the linear span of M if and only if $f(x) = 0$ for all $f \in X'$ such that $f = 0$ on M.*

Exercise 11. *Assume $x \in \ell^\infty$. Show that if the series $\sum_{i=1}^{\infty} x_i y_i$ is convergent for all $y \in \ell^2$, then $x \in \ell^2$.*

Exercise 12. *Assume $x \in \ell^\infty$. Prove that if the series $\sum_{i=1}^{\infty} x_i y_i$ is convergent for all $y \in c_0$, then $x \in \ell^1$.*

Exercise 13. *Suppose X is a Banach space and $(x_n)_{n\geq 1}$ is a sequence in X such that the sequence $(f(x_n))_{n\geq 1}$ is bounded for all $f \in X'$. Show that the sequence $(x_n)_{n\geq 1}$ is bounded.*

Exercise 14. *Are the following operators open?*

1. *$T : \mathbb{R}^2 \to \mathbb{R}$ defined by $Tx = x_1$ for all $x \in \mathbb{R}^2$;*

2. *$T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $Tx = (x_1, 0)$ for all $x \in \mathbb{R}^2$.*

Exercise 15. *Let X and Y be two Banach spaces, and $T : X \to Y$ a linear, continuous, and injective operator. Consider the operator $T^{-1} : R(T) \to X$. Show that T^{-1} is bounded if and only if $R(T)$ is a closed subspace of Y.*

Exercise 16. *Given X and Y as two normed vector spaces and $T : X \to Y$ a closed operator.*

1. *Prove that $T(K)$ is closed in Y for all compact subsets K in X.*

2. *Prove that $T^{-1}(K)$ is closed in X for all compact subsets K in Y.*

Exercise 17. *Show that if $T : X \to Y$ is a closed operator, where X and Y are normed vector spaces and Y is compact, then T is bounded.*

Exercise 18. *Let X and Y be two normed spaces, with X being compact. Prove that if $T : X \to Y$ is a closed and onto and one-to one (bijective) operator, then T^{-1} is bounded.*

Exercise 19. *Prove that if $T : X \to Y$ is a closed operator, where X and Y are normed vector spaces, then the kernel of $T(\ker(T))$ is closed.*

Exercise 20. *Let X and Y be two normed spaces, and $S : X \to Y$ is a closed operator, and $T : X \to Y$ is a bounded operator. Show that $S + T$ is bounded.*

Exercise 21. *Show that a normed vector space E that admits a countable basis cannot be complete.*

Exercise 22. *For a given vector space E, if any norm on E makes it a Banach space, what can be said about E?*

Exercise 23. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function, and suppose that for every $x > 0$, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to 0. Prove that the limit of f as x approaches $+\infty$ is 0.*