Module: CAF

TD1 Metric spaces - Normed vector spaces

Exercise 1. *Let* $\ell^\infty = \{x = (\xi_i)\, i = 1^\infty, \xi_i \in \mathbb{R}$ *s. t.* $\sup\limits_i |\xi_i| < \infty\}$. *It is recalled that* ℓ^∞ *equipped with the norm* $\|.\|_\infty$ *is a Banach space. We consider the following sets:*

$$c = \text{set of real convergent sequences}, \quad (1)$$
$$c_0 = \text{set of real sequences converging to 0}, \quad (2)$$
$$c_{00} = \text{set of real sequences with a finite number of nonzero terms}. \quad (3)$$

1. *Verify that* c, c_0, *and* c_{00} *are vector subspaces of* ℓ^∞.
2. *Are the spaces* c, c_0, *and* c_{00} *Banach spaces? Justify.*
3. *Show that* c_{00} *is dense in* c_0.

Exercise 2. *Given a normed vector space* $(X, \|\cdot\|)$. *The set* $e_n, n \geq 1$ *is called a* ***Schauder*** *basis for* X *if for every* $x \in X$, *there exist unique* $\alpha_n \in \mathbb{K}$ *such that* $\|x - \sum_{i=1}^n \alpha_i e_i\| \to 0$ *as* $n \to +\infty$. *The series* $\sum_{i=1}^\infty \alpha_i e_i$ *is called the expansion of* x *with respect to the basis* $e_n, n \geq 1$, *and we write* $x = \sum_{i=1}^\infty \alpha_i e_i$.

1. *Show that* $e_n = (\delta_{in})_{i\geq 1}, n \geq 1$ *is a Schauder basis in* ℓ^p, *where* $p \geq 1$.
2. *Show that if* X *has a Schauder basis, then* X *is separable.*
3. *Conclude that* ℓ^∞ *does not have a Schauder basis.*

Exercise 3.

1. *Construct the unit balls in* $(\mathbb{R}^2, \|.\|_p)$ *corresponding to* $p = 1, 2, \infty$.
2. *Show that* $N(\xi_1, \xi_2) = \sqrt{\xi_1^2 + 2\xi_1\xi_2 + 5\xi_2^2}$, $\xi_1, \xi_2 \in \mathbb{R}$ *defines a norm on* $\mathbb{R}^2$. *Construct its unit ball.*

Exercise 4. *Let* X *be a vector space, and let* $N : X \to \mathbb{R}$ *be a function satisfying the following properties for all* $x \in X$ *and* $\alpha \in \mathbb{R}$:

- $N(x) \geq 0$, *and* $N(x) = 0$ *if and only if* $x = 0$.
- $N(\alpha x) = |\alpha| N(x)$.

Show the equivalence of the following statements:

1. N *is a norm on* X.
2. N *is convex.*
3. *The set* $\{x \in X \mid N(x) \leq 1\}$ *is convex.*

Exercise 5. *Let* Y *be a closed subspace of a normed vector space* $(X, \|.\|)$.

1. *Show that the function* $\|.\|_0 : X/Y \to \mathbb{R}$ *defined by* $\|\hat{x}\|_0 = \inf\{\|x\| \mid x \in \hat{x}\}$ *defines a norm on* X/Y.

2. *Prove that if X is complete, then X/Y is a Banach space.*

Exercise 6. *Let C be a compact convex set in $\mathbb{R}^n$, such that 0 belongs to the interior of this compact convex set. The gauge of this convex set (or the Minkowski functional of the convex set) is the function $j : \mathbb{R}^n \to \mathbb{R}_+$ defined as*

$$j(x) = \inf\{t > 0 \mid \frac{x}{t} \in C\}. \tag{4}$$

Show that for all $x, y \in \mathbb{R}^n$ and $\alpha > 0$:

1. $j(x) = 0$ *if and only if* $x = 0$.
2. $j(\alpha x) = \alpha j(x)$.
3. $j(x + y) \leq j(x) + j(y)$.
4. *If C is symmetric with respect to the origin, the gauge j defines a norm on $\mathbb{R}^n$ whose closed unit ball is the compact set C.*

Exercise 7. *Let X be a normed vector space over $\mathbb{R}$. X is called **uniformly convex** if for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$\|x\| = \|y\| = 1 \text{ and } \|x - y\| \geq \varepsilon \text{ imply } \left\|\frac{x+y}{2}\right\| \leq 1 - \delta(\varepsilon). \tag{5}$$

1. *Verify that $(\mathbb{R}^n, \|.\|_p)$, for $1 < p < \infty$, are uniformly convex.*
2. *Are the spaces $(\mathbb{R}^n, \|.\|_1)$ and $(\mathbb{R}^n, \|.\|_\infty)$ uniformly convex?*

Exercise 8. *Given X as a Banach space and K as a compact subset in X.*

1. *Show that for every $x \in X$, there exists $y \in K$ such that $\|x - y\| = \inf_{z \in K} \|x - z\|$.*
2. *Show that if y is unique, then the map $P_K : X \to K$ defined by $\|x - P_K x\| = \inf_{z \in K} \|x - z\|$ is continuous.*

Exercise 9. *Let Y be a finite-dimensional subspace of a normed vector space X.*

1. *Show that for every $x \in X$, there exists $y \in Y$ such that $\|x - y\| = \inf_{z \in Y} \|x - z\|$.*
2. *Show that if, in addition, X is uniformly convex, then y is unique.*

Exercise 10. *Show that the interior of a compact subset in an infinite-dimensional normed vector space is empty.*

Exercise 11. *Let X be a vector space, $p : X \to \mathbb{R}$ be a seminorm, and $Y = \{x \in X \mid p(x) = 0\}$. Consider $X/Y = \{\hat{x} = x + Y \mid x \in X\}$, the associated quotient space (composed of equivalence classes resulting from the equivalence relation $x_1 \sim x_2$ if $x_1 - x_2 \in Y$).*

1. *Verify that Y is a subspace of X.*
2. *Show that $p(0) = 0$ and $|p(x) - p(y)| \leq p(x - y)$ for all $x, y \in X$.*
3. *Conclude that $\|\hat{x}\|_0 = p(x)$, where $x \in \hat{x}$, is uniquely defined, and $\|.\|_0 : X/Y \to \mathbb{R}$, $\|\hat{x}\|_0 = p(x)$, $x \in \hat{x}$ defines a norm on the quotient space X/Y.*

Exercise 12. *Consider X, the vector space of C^1 functions on $[0,1]$ such that $f(0)=0$.*

1. *Define, for $f \in X$, $N(f) = \|f\|_\infty + \|f'\|_\infty$. Show that N is a norm on X.*
2. *Show that for all $f \in X$ and $x \in [0,1]$, $f(x) = e^{-x}\int_0^x e^t(f(t)+f'(t))\,dt$.*
3. *Define, for $f \in X$, $N'(f) = \|f+f'\|_\infty$. Show that N' is a norm on X and is equivalent to N.*

Exercise 13. *Let A be the set of continuous functions on $[0,1]$ such that $f(x) \geq 0$ for all $x \in [0,1]$.*

1. *Equip $C[0,1]$ with the norm $\|f\|_\infty = \sup_{x\in[0,1]}|f(x)|$. Show that A is closed and determine its interior.*
2. *Equip $C[0,1]$ with the norm $\|f\|_1 = \int_0^1 |f(x)|\,dx$. Show that the interior of A is empty and that A is closed.*

Exercise 14. *Consider $X = C^1([a,b])$.*

1. *Is X complete when equipped with the uniform norm $\|.\|_\infty$?*
2. *Consider the norm $N(f) = \sup_{t\in[a,b]}\|f(t)\| + \sup_{t\in[a,b]}\|f'(t)\|$. Is the space (X,N) complete?*

Exercise 15. *Let $A = \{x_n = (\delta_{in})_{i=1}^\infty \mid n = 1,2,\dots\} \subset \ell^\infty$.*

1. *Verify that*

$$\|x_n\|_\infty = 1, \text{ for all } n \geq 1, \tag{6}$$
$$\|x_n - x_m\|_\infty = 1 - \delta_{nm}, \text{ for all } n, m \geq 1. \tag{7}$$

2. *Conclude that A is closed and bounded.*
3. *Is A compact? Justify your answer.*

Exercise 16. *Let X be a uniform convex Banach space, and let K be a non-empty, convex, closed subset in X. Show that for every $x \in X$, there exists a unique $y \in K$ such that $\|x-y\| = \inf_{z\in K}\|x-z\|$.*