Kasdi Merbah Ouargla University
Mathematics Department 2024-2025 academic year

Exercise set 1 - Asymptotic Analysis

Exercise 1: Consider the following functions:

$$\begin{aligned}(\mathbf{a})\quad f(\varepsilon) &= \sqrt{1+\varepsilon^2}, \quad (\mathbf{b})\quad f(\varepsilon)=\varepsilon\sin(\varepsilon), \quad (\mathbf{c})\quad f(\varepsilon)=(1-e^{\varepsilon})^{-1}\\ (\mathbf{d})\quad f(\varepsilon) &= \ln(1+\varepsilon), \quad (\mathbf{e})\quad f(\varepsilon)=\varepsilon\ln\varepsilon.\end{aligned}$$

(1) For which values of α do we have $f = O(\varepsilon^\alpha)$ as $\varepsilon \downarrow 0$?
(2) For which values of α do we have $f = o(\varepsilon^\alpha)$ as $\varepsilon \downarrow 0$?

Exercise 2: Consider the function $\Phi(x,\varepsilon) = e^{-x/\varepsilon}$ defined on the domain $D = \{x : x \in [0,1]\}$. Determine the order of magnitude of the function Φ, that is, find the real number t such that $\Phi = 0_s(\varepsilon^t)$ on D for the following norms:
(a) $\|\Phi\| = \sup_{x\in D} |\Phi(x,\varepsilon)|$, (b) $\|\Phi\| = \sup_{x\in D} |\Phi(x,\varepsilon)| + \sup_{x\in D} \left|\frac{d}{dx}\Phi(x,\varepsilon)\right|$,(c) $\|\Phi\| = \left[\int_0^1 (\Phi(x,\varepsilon))^2\, dx\right]^{\frac{1}{2}}$,
(d) $\|\Phi\| = \left[\int_0^1 (\Phi(x,\varepsilon))^2\, dx\right]^{\frac{1}{2}} + \left[\int_0^1 \left(\frac{d}{dx}\Phi(x,\varepsilon)\right)^2 dx\right]^{\frac{1}{2}}$.

Exercise 3: Determine which of the following are **asymptotic sequences**:

$$(a) \qquad \log(1+\varepsilon^n), \qquad n=0,1,\ldots, \qquad \varepsilon \longrightarrow 0.$$

$$(b) \qquad \varepsilon^{a_n} e^{-n/\varepsilon}, \qquad n=0,1,\ldots, \qquad \varepsilon \longrightarrow 0, \text{ where } a_{n+1} > a_n \quad \forall n \geq 0.$$

$$(c) \qquad \varepsilon^n \left[a + \cos(\varepsilon^{-n})\right],, \qquad n=0,1,\ldots, \qquad \varepsilon \longrightarrow 0, \text{ where } a > 1.$$

$$(d) \qquad \frac{d}{d\varepsilon}\left\{\varepsilon^n \left[a + \cos(\varepsilon^{-n})\right]\right\},, \qquad n=0,1,\ldots, \qquad \varepsilon \longrightarrow 0, \text{ where } a > 1.$$

Exercise 4: Consider the function

$$\phi(\varepsilon) = \exp(-1/\varepsilon), \quad \text{for} \quad \varepsilon > 0.$$

Show that, in an asymptotic expansion of the form

$$\phi(\varepsilon) \sim a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \cdots ,$$

valid as $\varepsilon \longrightarrow 0^+$, all the coefficients a_0, a_1, $a_2, \ldots$ are zero.

Exercise 5: Show that

$$F(\varepsilon) = \int_0^\infty \frac{e^{-t}}{(1+\varepsilon t)^2} dt \backsim \sum_{n\geq 0} (-1)^n (n+1)! \varepsilon^n \quad \text{as} \quad \varepsilon \longrightarrow 0^+.$$

Obtain an estimate of $F(0.1)$ by 'optimal' truncation of this asymptotic expansion.

Exercise 6: Let $G(x)$ be defined for $x > 0$ by

$$G(x) = \int_0^\infty \frac{e^{-t}}{(t+x)} dt.$$

1- Show by integration by parts that

$$G(x) = \sum_{n=1}^{N} \frac{(-1)^{n-1}(n-1)!}{x^n} + (-1)^N N! \int_0^\infty \frac{e^{-t}}{(t+x)^{N+1}} dt.$$

2- Show that the absolute value of this last integral is at most $\frac{1}{x^{N+1}}$, and deduce that $G(x)$ has the asymptotic expansion

$$G(x) \backsim \sum_{n=1}^{N} \frac{(-1)^{n-1}(n-1)!}{x^n} \quad \text{as } x \longrightarrow \infty.$$

3- If x is allowed to be complex, show that the same method proves this asymptotic expansion in the sector $|arg(x)| \leq \pi - \delta$, for fixed $\delta > 0$.

Exercise 7: (1)- Show that the partial sums of the series $\sum_{k=1}^{\infty}(-1)^{k+1}k^{-1}$ satisfy

$$f_n \equiv \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} = \log 2 - (-1)^n \int_0^1 \frac{\xi^n}{1+\xi} d\xi.$$

(2)- By integration by part, or othewise, show that

$$\int_0^1 \frac{\xi^n}{1+\xi} d\xi \sim \sum_{k\geq 1} \frac{(k-1)!n!}{2^k(n+k)!} \quad \text{as } n \longrightarrow \infty.$$

(3)- Apply the **Shanks transform** to f_n to show that

$$Sf_n = f_n + \frac{(-1)^n}{2n+1} = \log 2 + \frac{(-1)^n}{8n^3} + O(n^{-4}).$$

(**Definition**: Given a sequence (f_n), the **Shanks transform** generates a new sequence (Sf_n) that converges faster to the same limit. The transformation is defined as:

$$Sf_n = f_{n+2} - \frac{\left(f_{n+2} - f_{n+1}\right)^2}{f_{n+2} - 2f_{n+1} + f_n}$$

provided that the denominator is nonzero.)

(4)- What accuracy might one except to obtain by evaluating f_5 and Sf_5 as approximation to $\log 2$?