Module:Optimization 1

–Tutorial 03 –

Exercise 01:

Given a symmetric matrix of size (n,n), a vector $\mathbf{x}$ in $\mathbb{R}^n$, and a real number c, consider the function f defined on $\mathbb{R}^n$ with values in $\mathbb{R}$ by:

$$f(\mathbf{x}) = \frac{1}{2}\langle A\mathbf{x}, \mathbf{x}\rangle + \langle \mathbf{b}, \mathbf{x}\rangle + c$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Denote by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ the eigenvalues of A.

Recall that:

- The matrix A is positive definite if and only if all its eigenvalues are positive.
- f is strictly convex in $\mathbb{R}^n$ if and only if for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $t \in (0,1)$:

$$\langle \nabla f(x) - f(y), x - y\rangle \geq 0$$

Show that:

1. Show that $\mathbf{x} \in \mathbb{R}^n, \lambda_1 \|x\|^2 \leq \langle Ax, x\rangle \leq \lambda_n \|x\|]$
2. f is strictly convex in $\mathbb{R}^n$ if and only if A is positive definite.
3. If A is positive definite, then $\lim_{\|x\|\to\infty} f(x) = \infty$

Exercise 02:

1. Is the composition of two convex functions convex?
2. Is the product of two convex functions convex?
3. Examine, among the following functions, which ones are convex, strictly convex:

$$f_1(x) = e^x, \quad f_2(x) = x^2, \quad f_3(x) = \log(x), \quad f_4(x) = \sqrt{x}$$

4. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Show that the functions

$$g_1(x) = \|x\|, \quad g_2(x) = \|x\|^2$$

are convex.

Exercise 03:

Let f be a function defined on $\mathbb{R}^n$ by

$$e^{x^2+y^2+z^2}$$

1. Show that f is strictly convex.