Subject:Optimization 1

–Tutorial 02 –

Exercise 1

Given a symmetric matrix of size (n,n), a vector $b \in \mathbb{R}^n$, and a real number c, we consider the function f defined on $\mathbb{R}^n$ with values in $\mathbb{R}$ as follows:

$$f(x) = \frac{1}{2}\langle Ax, x\rangle + \langle b, x\rangle + c,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$, and $\| \cdot \|$ is a norm on $\mathbb{R}^n$.

1. For all $x, y \in \mathbb{R}^n$ and $t \in [0,1]$, show that

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y) - \frac{1}{2}t(1-t)\langle A(x-y), x-y\rangle.$$

2. Show that the function f is strictly convex in $\mathbb{R}^n$ if and only if the matrix A is positive definite.

a) Show that the function g defined on $\mathbb{R}^n$ by $g(x) = \langle Ax, x\rangle$ has a global minimum point on the unit sphere for the Euclidean norm.

b) Deduce that the matrix A is positive definite if and only if there exists a positive real number α such that for all $x \in \mathbb{R}^n$, $\alpha\|x\|^2 \leq \langle Ax, x\rangle$.

3. Show that the function f is strictly convex in $\mathbb{R}^n$ if and only if there exists a positive real number α such that for all $x, y \in \mathbb{R}^n$ and $t \in [0,1]$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \frac{\alpha}{2}t(1-t)\|x-y\|^2.$$

4. Study the convexity of the following functions:
   (a) $f(x) = \ln x$ for $x > 0$.
   (b) $f(x) = x^q$ for $x \geq 0$, $0 \leq q \leq 1$.
   (c) $f(x,y) = x^2 + y^2$.
   (d) $f(x,y) = x^2 - 2y^2 + xy$.
   (e) $f(x,y,z) = x^2 + 2y^2 + 3z^2 - xy + 4yz$.

Exercise 2

Let C be a convex subset of $\mathbb{R}^n$. A function f on C with values in $\mathbb{R}^n$ is called affine if for all $x, y \in \mathbb{R}^n$ and $t \in [0,1]$,

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y).$$

a) If a is a vector in $\mathbb{R}^n$ and b is a real number, show that the function f defined on $\mathbb{R}^n$ by $f(x) = \langle a, x\rangle + b$ is affine.

b) Let f be an affine function on C, and g be a convex function on an interval I in $\mathbb{R}^n$ such that $f(C) \subset I$. Show that the composition $g \circ f$ is convex on C.

c) Deduce that the function defined by $h(x,y) = \ln(1 + 2x + 3y)$ is concave on its domain, which will be determined.