KASDI MERBAH UNIVERSITY-OUARGLA 2024-2025

2^{ND} YEAR MASTER : MOD. & NUM. ANAL. ASYMPTOTIC MODELING OF THIN PLATES

EXAM DURATION 01^h30^{mn}

EXERCISE 1. (08 PTS)

Let Ω be a domain in $\mathbb{R}^3$, let $\Gamma_0 \subset \partial\Omega$ with area $\Gamma_0 > 0$ and let

$$\mathbf{V}(\Omega) = \left\{ \mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega);\ \mathbf{v} = 0 \text{ on } \Gamma_0 \right\}.$$

1. Show that the semi-norm $|.|$ defined by
$$|\mathbf{v}| = |e(\mathbf{v})|_{0,\Omega}$$
becomes a norm over the space $\mathbf{V}(\Omega)$.
2. Show that there exists a constant $C(\Omega, \Gamma_0)$ such that
$$\|\mathbf{v}\|_{1,\Omega} \leq C(\Omega, \Gamma_0) |e(\mathbf{v})|_{0,\Omega} \text{ for all } \mathbf{v} \in \mathbf{V}(\Omega).$$

EXERCISE 2. (12 PTS)

We consider the unknown displacement $u^\varepsilon = (u_i^\varepsilon)$ satisfies the following boundary problem

$$\begin{cases} -\partial_j^\varepsilon \{\lambda^\varepsilon e_{pp}^\varepsilon(u^\varepsilon)\delta_{ij} + 2\mu^\varepsilon e_{ij}^\varepsilon(u^\varepsilon)\} = f_i^\varepsilon \text{ in } \Omega^\varepsilon, \\ \mathrm{u}_i^\varepsilon = 0 \text{ on } \Gamma_0^\varepsilon, \\ \{\lambda^\varepsilon e_{pp}^\varepsilon(u^\varepsilon)\delta_{ij} + 2\mu^\varepsilon e_{ij}^\varepsilon(u^\varepsilon)\} n_j^\varepsilon = \begin{cases} g_i^\varepsilon \text{ on } \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon, \\ 0 \text{ on } \Gamma_1^\varepsilon, \end{cases} \end{cases} \tag{1}$$

where $e_{ij}^\varepsilon(u^\varepsilon) = \frac{1}{2}(\partial_i^\varepsilon u_j^\varepsilon + \partial_j^\varepsilon u_i^\varepsilon)$, $f_i^\varepsilon \in L^2(\Omega^\varepsilon)$ and $g_i^\varepsilon \in L^2(\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon)$.

1. Find the variational formulation of (1) in the space $V(\Omega^\varepsilon)$.
2. Write the variational formulation obtained in the following form $B^\varepsilon(u^\varepsilon, v^\varepsilon) = L^\varepsilon(v^\varepsilon)$.
3. With each point $x \in \bar{\Omega}$, we associate the point $x^\varepsilon \in \bar{\Omega}^\varepsilon$ through the bijection
$\pi^\varepsilon : x = (x_1, x_2, x_3) \in \bar{\Omega} \to x^\varepsilon = (x_1, x_2, \varepsilon x_3) \in \bar{\Omega}^\varepsilon$.
With the functions u^ε, $v^\varepsilon \in V(\Omega^\varepsilon)$ we associate the functions $u(\varepsilon)$, v defined by
$$\begin{cases} u_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 u_\alpha(\varepsilon)(x), u_3^\varepsilon(x^\varepsilon) = \varepsilon u_3(\varepsilon)(x), \forall x^\varepsilon = \pi^\varepsilon x \in \bar{\Omega}^\varepsilon, \\ v_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 v_\alpha(x), v_3^\varepsilon(x^\varepsilon) = \varepsilon v_3(x), \forall x^\varepsilon = \pi^\varepsilon x \in \bar{\Omega}^\varepsilon. \end{cases}$$
We make the following assumptions on the data
$$\begin{cases} \lambda^\varepsilon = \lambda, \mu^\varepsilon = \mu, f_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 f_\alpha(x), f_3^\varepsilon(x^\varepsilon) = \varepsilon^3 f_3(x), \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \\ g_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^3 g_\alpha(x), g_3^\varepsilon(x^\varepsilon) = \varepsilon^4 g_3(x), \forall x^\varepsilon = \pi^\varepsilon x \in \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon, \end{cases}$$
where the constants $\lambda > 0$, $\mu > 0$ and the functions $f_i \in L^2(\Omega)$, $g_i \in L^2(\Gamma_+ \cup \Gamma_-)$ are independent of ε.

3.1. In the space $V(\Omega)$, formulate the equivalent "scaled" problem in the following form :
$$\frac{1}{\varepsilon^4} B_{-4}(u(\varepsilon), v) + \frac{1}{\varepsilon^2} B_{-2}(u(\varepsilon), v) + B_0(u(\varepsilon), v) = L(v). \tag{2}$$

3.2. Show that the variational formulation (2) takes the following form :
$$\int_\Omega A\kappa(\varepsilon) : \kappa(\varepsilon; v) dx = L(v),$$
where $AB : C = \lambda b_{pp} c_{qq} + 2\mu b_{ij} c_{ij}$, $\kappa_{\alpha\beta}(\varepsilon; v) = e_{\alpha\beta}(v)$, $\kappa_{\alpha 3}(\varepsilon; v) = \frac{1}{\varepsilon} e_{\alpha 3}(v)$, $\kappa_{33}(\varepsilon; v) = \frac{1}{\varepsilon^2} e_{33}(v)$,
$\kappa(\varepsilon) = \kappa(\varepsilon; u(\varepsilon))$, $e_{ij}(v) = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$.

3.3. Show that
$$u(\varepsilon) \rightharpoonup u \text{ in } \mathbf{H}^1(\Omega), \varepsilon \to 0,$$
$$\kappa(\varepsilon) \rightharpoonup \kappa \text{ in } \mathbf{L}_s^2(\Omega), \varepsilon \to 0.$$

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<u>Cor. 1</u>

1. We have $\partial_j(\partial_k v_i) = \partial_j e_{ik}(v) + \partial_k e_{ij}(v) - \partial_i e_{jk}(v)$ in $\mathcal{D}'(\Omega)$. (0,5)

Therefore $|e(v)|_{0,\Omega} = 0 \Rightarrow \partial_j(\partial_k v_i) = 0$ in $\mathcal{D}'(\Omega)$. (0,5)

By a classical result from distribution theory, shows that there exist constants c_i and b_{ij} such that

$$v_i(x) = c_i + \sum_{j=1}^{3} b_{ij}\, x_j \qquad \forall\, x = (x_i) \in \Omega. \quad (0,5)$$

But $e_{ij}(v) = 0$ also implies that $b_{ij} = -b_{ji}$. (0,5)

Hence there exist $c, d \in \mathbb{R}^3$ such that (0,5)

$$v(x) = c + d \wedge ox \qquad \forall\, x \in \Omega.$$

Since area $\Gamma_0 > 0$, thus $c = d = 0$. Then $v = 0$. (0,5)

2. If this inequality is false, there exists a sequence $(v^k)_{k=1}^{\infty}$ of elements $v^k \in V(\Omega)$ such that (0,5)

$$\|v^k\|_{1,\Omega} = 1 \ \ \forall k \quad \text{and} \quad \lim_{k\to\infty} |e(v^k)|_{0,\Omega} = 0. \quad (0,5)$$

Hence $(v^k)_{k=1}^{\infty}$ is bounded in $H^1(\Omega)$, there exists a subsequence $(v^l)_{l=1}^{\infty}$ that converges in $L^2(\Omega)$ by the Rellich-Kondrašov theorem. (0,5)

Also $(e(v^l))_{l=1}^{\infty}$ converges in $L^2(\Omega)$.

Therefore $(v^l)_{l=1}^{\infty}$ is a Cauchy sequence with respect to the norm $\|\cdot\|$ defined by

$$\|v\| = \left\{ |v|_{0,\Omega}^2 + |e(v)|_{0,\Omega}^2 \right\}^{1/2}. \quad (0,5)$$

According to Korn's inequality without boundary conditions, we obtain

$$\|v\|_{1,\Omega} \leq C(\Omega)\, \|v\|. \quad (0,5)$$

Then $(v^l)_{l=1}^{\infty}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{1,\Omega}$. (0,5)

- The space $V(\Omega)$ being complete, as a closed subspace of $H^1(\Omega)$, there exists $v \in V(\Omega)$ such that (1)

$$v^l \to v \text{ in } H^1(\Omega).$$

- Since $|e(v)|_{0,\Omega} = \lim_{l\to\infty} |e(v^l)|_{0,\Omega} = 0$, hence $v = 0$ by the question 1. But this contradicts the relation $\|v^l\|_{1,\Omega} = 1$ $\forall l$. (1)

Cor. 2

1. $V(\Omega^\varepsilon) = \{ v^\varepsilon = (v_i^\varepsilon) \in H^1(\Omega^\varepsilon) ; v^\varepsilon = 0 \text{ on } \Gamma_0^\varepsilon \}$. 0,5

Using Green formula $\int_\Omega (\partial_j w) v \, dx = -\int_\Omega w \, \partial_j v \, dx + \int_\Gamma w v n_j \, d\Gamma$, we obtain

$$-\int_{\Omega^\varepsilon} \partial_j^\varepsilon \{\dots\} v_i^\varepsilon \, dx^\varepsilon = \int_{\Omega^\varepsilon} \{\dots\} \partial_j^\varepsilon v_i^\varepsilon \, dx^\varepsilon - \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} \{\dots\} v_i^\varepsilon n_j^\varepsilon \, d\Gamma^\varepsilon = \int_{\Omega^\varepsilon} f_i^\varepsilon v_i^\varepsilon \, dx^\varepsilon$$ 0,5

Since $e_{ij}^\varepsilon(u^\varepsilon) \partial_j^\varepsilon v_i^\varepsilon = e_{ij}^\varepsilon(u^\varepsilon) e_{ij}^\varepsilon(v^\varepsilon)$, 0,5 we have

$$\int_{\Omega^\varepsilon} \{ \lambda^\varepsilon e_{pp}^\varepsilon(u^\varepsilon) e_{qq}^\varepsilon(v^\varepsilon) + 2\mu^\varepsilon e_{ij}^\varepsilon(u^\varepsilon) e_{ij}^\varepsilon(v^\varepsilon) \} dx^\varepsilon = \int_{\Omega^\varepsilon} f_i^\varepsilon v_i^\varepsilon \, dx^\varepsilon + \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} g_i^\varepsilon v_i^\varepsilon \, d\Gamma^\varepsilon.$$ 0,5

2. $B^\varepsilon(u^\varepsilon, v^\varepsilon) = \int_{\Omega^\varepsilon} \{\dots\} dx^\varepsilon$, 0,5 $L^\varepsilon(v^\varepsilon) = \int_{\Omega^\varepsilon} f_i^\varepsilon v_i^\varepsilon \, dx^\varepsilon + \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} g_i^\varepsilon v_i^\varepsilon \, d\Gamma^\varepsilon$ 0,5

0,5 0,5

3.1. Using the bijection π^ε, we obtain $u(\varepsilon) \in V(\Omega) = \{ v = (v_i) \in H^1(\Omega) ; v = 0 \text{ on } \Gamma_0 \}$.

Using $\partial_\alpha^\varepsilon = \partial_\alpha$, 0,5 $\partial_3^\varepsilon = \frac{1}{\varepsilon}\partial_3$, $\int_{\Omega^\varepsilon} \theta(x^\varepsilon) dx^\varepsilon = \varepsilon \int_\Omega \theta(\pi^\varepsilon x) dx$, $\int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} \theta(x^\varepsilon) d\Gamma^\varepsilon = \int_{\Gamma_+ \cup \Gamma_-} \theta(\pi^\varepsilon x) d\Gamma$, 0,5

The scalings and the assumptions, we obtain

$$\frac{1}{\varepsilon^4} \int_\Omega (\lambda + 2\mu) e_{33}(u(\varepsilon)) e_{33}(v) dx + \frac{1}{\varepsilon^2} \int_\Omega \{ \lambda e_{\sigma\sigma}(u(\varepsilon)) e_{33}(v) + \lambda e_{33}(u(\varepsilon)) e_{\tau\tau}(v) + 4\mu e_{\alpha 3}(u(\varepsilon)) e_{\alpha 3}(v) \} dx$$

$$+ \int_\Omega \{ \lambda e_{\sigma\sigma}(u(\varepsilon)) e_{\tau\tau}(v) + 2\mu e_{\alpha\beta}(u(\varepsilon)) e_{\alpha\beta}(v) \} dx = \int_\Omega f_i v_i \, dx + \int_{\Gamma_+ \cup \Gamma_-} g_i v_i \, d\Gamma.$$ 1

Then $B_{-4}(u(\varepsilon), v) = \dots$, 0,5 $B_{-2}(u(\varepsilon), v) = \dots$ $B_0(u(\varepsilon), v) = \dots$, $L(v) = \dots$.

3.2 We have

$$\frac{1}{\varepsilon^4} B_{-4}(u(\varepsilon), v) + \frac{1}{\varepsilon^2} B_{-2}(u(\varepsilon), v) + B_0(u(\varepsilon), v) = \int_\Omega \Big[\lambda \{ e_{\sigma\sigma} e_{\tau\tau} + e_{\sigma\sigma} \frac{1}{\varepsilon^2} e_{33} + \frac{1}{\varepsilon^2} e_{33} e_{\tau\tau} + \frac{1}{\varepsilon^2} e_{33} \frac{1}{\varepsilon^2} e_{33} \}$$

$$+ 2\mu \{ e_{\alpha\beta} e_{\alpha\beta} + \frac{1}{\varepsilon} e_{\alpha 3} \frac{1}{\varepsilon} e_{\alpha 3} + \frac{1}{\varepsilon} e_{3\alpha} \frac{1}{\varepsilon} e_{3\alpha} + \frac{1}{\varepsilon^2} e_{33} \frac{1}{\varepsilon^2} e_{33} \} \Big] dx$$

$$= \int_\Omega \Big[\lambda \{ \kappa_{\sigma\sigma}(\varepsilon) \kappa_{\tau\tau}(\varepsilon; v) + \kappa_{\sigma\sigma}(\varepsilon) \kappa_{33}(\varepsilon; v) + \kappa_{33}(\varepsilon) \kappa_{\tau\tau}(\varepsilon; v) + \kappa_{33}(\varepsilon) \kappa_{33}(\varepsilon; v) \}$$

$$+ 2\mu \{ \kappa_{\alpha\beta}(\varepsilon) \kappa_{\alpha\beta}(\varepsilon; v) + \kappa_{\alpha 3}(\varepsilon) \kappa_{\alpha 3}(\varepsilon; v) + \kappa_{3\alpha}(\varepsilon) \kappa_{3\alpha}(\varepsilon; v) + \kappa_{33}(\varepsilon) \kappa_{33}(\varepsilon; v) \} \Big] dx$$ 2

$$= \int_\Omega A\kappa(\varepsilon) : \kappa(\varepsilon; v) \, dx.$$

then $\int_\Omega A\kappa(\varepsilon) : \kappa(\varepsilon; v) dx = L(v)$.

3.3 Let $v = u(\varepsilon)$ in the previous variational equation, we obtain

$$\int_\Omega A\kappa(\varepsilon) : \kappa(\varepsilon) dx = L(u(\varepsilon)).$$ 0,5

since $2\mu \, b_{ij} b_{ij} \leq AB : B$ 0,5 and we assume $\varepsilon \leq 1$, 0,5 we obtain using Korn's inequality with boundary conditions: 0,5

$$
\begin{aligned}
2\mu\, C(\Omega,\Gamma_0)^{-2}\, \|u(\varepsilon)\|^2_{1,\Omega} &\leq 2\mu\, |e(u(\varepsilon))|^2_{0,\Omega} \\
&\leq 2\mu\, |K(\varepsilon)|^2_{0,\Omega} \\
&\leq \int_\Omega A K(\varepsilon) : K(\varepsilon)\, dx \\
&\leq \|L\|_{\mathcal{L}(H^1(\Omega);\mathbb{R})} \|u(\varepsilon)\|_{1,\Omega}.
\end{aligned}
$$

Then the norms $\|u(\varepsilon)\|_{1,\Omega}$ and consequently the norms $|K(\varepsilon)|_{0,\Omega}$ are bounded independently of ε.

Hence there exists a subsequence $(u(\varepsilon))$ and a subsequence $(K(\varepsilon))$, and there exist $u \in H^1(\Omega)$ and $k \in L^2_s(\Omega)$ such that

$$
\begin{aligned}
u(\varepsilon) &\longrightarrow u \text{ in } H^1(\Omega) \text{ as } \varepsilon \to 0, \\
K(\varepsilon) &\longrightarrow k \text{ in } L^2_s(\Omega) \quad \text{as } \varepsilon \to 0.
\end{aligned}
$$