2ND YEAR MASTER : MOD. & NUM. ANAL. DIFFERENTIAL GEOMETRY AND ELASTICITY

EXAM DURATION $01^{h}30^{mn}$

EXERCISE 1. (05 PTS)

Let $\Theta : \Omega \to E^3$ be an injective immersion and C^2-diffeomorphism of Ω on $\widehat{\Omega} = \Theta(\Omega)$ (Ω an open set of $\mathbb{R}^3$ and E^3 a three-dimensional Euclidean space).
Given a vector field $v_i g^i : \Omega \to \mathbb{R}^3$ with $v_i \in C^1(\Omega)$.

1. Using $\partial_j g^i = -\Gamma^i_{jk} g^k$, show that
$$\partial_j g_q = \Gamma^p_{jq} g_p.$$
2. Show that
$$\partial_j (v_i g^i) = v_{i\|j} g^i,$$
where $v_{i\|j} = \partial_j v_i - \Gamma^p_{ij} v_p$.

EXERCISE 2. (07 PTS)

Let $\theta \in C^3(\omega; E^3)$ be an immersion (ω an open set of $\mathbb{R}^2$).

1. Show that
$$\partial_\alpha a_3 = -b_{\alpha\sigma} a^\sigma,$$
$$\partial_\alpha a_\beta = \Gamma^\sigma_{\alpha\beta} a_\sigma + b_{\alpha\beta} a_3,$$
where $b_{\alpha\sigma} = \partial_\alpha a_\sigma . a_3 = -\partial_\alpha a_3 . a_\sigma, \quad \Gamma^\sigma_{\alpha\beta} = a^\sigma . \partial_\alpha a_\beta$.
2. Show that
$$\partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + \Gamma^\mu_{\alpha\sigma} b_{\beta\mu} - \Gamma^\mu_{\alpha\beta} b_{\sigma\mu} = 0 \text{ in } \omega.$$

EXERCISE 3. (08 PTS)

Let Ω be a domain in $\mathbb{R}^3$ and let the space
$$\mathbf{W}(\Omega) = \left\{ \mathbf{v} = (v_i) \in \mathbf{L}^2(\Omega); e_{ij}(\mathbf{v}) \in L^2(\Omega) \right\},$$
where $e_{ij}(v) = \left\{ \frac{1}{2}(\partial_j v_i + \partial_i v_j) - \Gamma^p_{ij} v_p \right\}$, $\Gamma^p_{ij} = g^p . \partial_i g_j$.

1. Show that the space $\mathbf{W}(\Omega)$ equipped with the following norm
$$\|\mathbf{v}\| = \left\{ \sum_i \|v_i\|^2_{0,\Omega} + \sum_{i,j} \|e_{ij}(\mathbf{v})\|^2_{0,\Omega} \right\}^{1/2}$$
is a Hilbert space.
2. Show that $\mathbf{H}^1(\Omega) = \mathbf{W}(\Omega)$.
3. Show that there exists a constant $C(\Omega)$ such that
$$\|\mathbf{v}\|_{1,\Omega} \le C(\Omega) \|\mathbf{v}\| \text{ for all } v \in \mathbf{H}^1(\Omega).$$

Exam: GDE 2024–2025

Cor. 1

1. Since $g^p \cdot g_q = \delta^p_q$, we have $0 = \partial_j (g^p \cdot g_q) = \partial_j g^p \cdot g_q + g^p \cdot \partial_j g_q$

Using $\partial_j g^p = -\Gamma^p_{ji} g^i$, we obtain $0 = -\Gamma^p_{ji} g^i \cdot g_q + g^p \cdot \partial_j g_q$

thus $0 = -\Gamma^p_{jq} + g^p \cdot \partial_j g_q$, since $g^i \cdot g_q = \delta^i_q$.

therefore $g^p \cdot \partial_j g_q = \Gamma^p_{jq}$

Since $g^p \cdot g_q = 1$, then $\partial_j g_q = \Gamma^p_{jq} g_p$.

2. We have

$$\partial_j (v_i g^i) = (\partial_j v_i) g^i + v_i \partial_j g^i.$$

Using $\partial_j g^i = -\Gamma^i_{jk} g^k$, we obtain

$$\partial_j (v_i g^i) = (\partial_j v_i) g^i - v_i \Gamma^i_{jk} g^k$$
$$= (\partial_j v_i) g^i - \Gamma^p_{ij} v_p g^i$$
$$= (\partial_j v_i - \Gamma^p_{ij} v_p) g^i$$
$$= v_{i\|j} g^i.$$

Cor. 2

1. We have $\partial_\alpha a_3 \cdot a_3 = \frac{1}{2} (\partial_\alpha (a_3 \cdot a_3)) = 0$, then $\partial_\alpha a_3$ is in tangent plane.

Thus $\partial_\alpha a_3 = (\partial_\alpha a_3 \cdot a_\sigma) a^\sigma = -b_{\alpha\sigma} a^\sigma$.

$$\partial_\alpha a_\beta = (\partial_\alpha a_\beta \cdot a^\sigma) a_\sigma + (\partial_\alpha a_\beta \cdot a^3) a_3$$
$$= \Gamma^\sigma_{\alpha\beta} a_\sigma + b_{\alpha\beta} a_3, \text{ (since } a^3 = a_3).$$

2. We have
$$\partial_\alpha a_\beta \cdot \partial_\sigma a_3 = (\Gamma^\mu_{\alpha\beta} a_\mu + b_{\alpha\beta} a_3)(-b_{\sigma\tau} a^\tau)$$
$$= -\Gamma^\mu_{\alpha\beta} a_\mu b_{\sigma\tau} a^\tau - b_{\alpha\beta} b_{\sigma\tau} \underbrace{a_3 a^\tau}_{0}$$
$$= -\Gamma^\mu_{\alpha\beta} b_{\sigma\tau} a_\mu a^\tau$$
$$= -\Gamma^\mu_{\alpha\beta} b_{\sigma\tau} \delta^\tau_\mu = -\Gamma^\mu_{\alpha\beta} b_{\sigma\mu}.$$

$\partial_\sigma b_{\alpha\beta} = \partial_\sigma (\partial_\alpha a_\beta \cdot a_3) = \partial_{\alpha\sigma} a_\beta \cdot a_3 + \partial_\alpha a_\beta \cdot \partial_\sigma a_3$,

thus $\partial_{\alpha\sigma} a_\beta \cdot a_3 = \partial_\sigma b_{\alpha\beta} - \partial_\alpha a_\beta \cdot \partial_\sigma a_3 = \partial_\sigma b_{\alpha\beta} + \Gamma^\mu_{\alpha\beta} b_{\sigma\mu}$.

Similar, we obtain $\partial_{\alpha\beta} a_\sigma \cdot a_3 = \partial_\beta b_{\alpha\sigma} + \Gamma^\mu_{\alpha\sigma} b_{\beta\mu}$.

since $\partial_{\alpha\sigma} a_\beta = \partial_{\alpha\beta} a_\sigma$, thus $\partial_{\alpha\sigma} a_\beta \cdot a_3 = \partial_{\alpha\beta} a_\sigma \cdot a_3$. 0,5

then $\partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + \Gamma^\mu_{\alpha\sigma} b_{\beta\mu} - \Gamma^\mu_{\alpha\beta} b_{\sigma\mu} = 0$ in ω. 0,5

Cor. 3

1. We have $\int_\Omega e_{ij}(v)\varphi dx = -\int_\Omega \left\{ \frac{1}{2}(v_i \partial_j \varphi + v_j \partial_i \varphi) + \Gamma^p_{ij} v_p \varphi \right\} dx$, $\forall \varphi \in \mathcal{D}(\Omega)$. 0,5

Let $(v^k)_{k=1}^\infty$ with $v^k = (v_i^k) \in W(\Omega)$ be given a Cauchy sequence.
The definition of the norm $\|.\|$ shows that there exist functions $v_i \in L^2(\Omega)$ and $e_{ij} \in L^2(\Omega)$ such that 0,5

$v_i^k \to v_i$ in $L^2(\Omega)$ and $e_{ij}(v^k) \to e_{ij}$ in $L^2(\Omega)$ as $k \to +\infty$, 0,5

since $L^2(\Omega)$ is complete. 0,5
We obtain

$$\int_\Omega e_{ij}(v^k)\varphi dx = -\int_\Omega \left\{ \frac{1}{2}(v_i^k \partial_j \varphi + v_j^k \partial_i \varphi) + \Gamma^p_{ij} v_p^k \varphi \right\} dx, \quad k \geq 1.$$ 0,5

Letting $k \to +\infty$, we obtain $e_{ij} = e_{ij}(v)$. 0,5

Then $W(\Omega)$ is a Banach by $\|.\|$, thus $W(\Omega)$ is a Hilbert. 0,5

2. Clearly $H^1(\Omega) \subset W(\Omega)$. 0,5

Let $v = (v_i) \in W(\Omega)$, then $s_{ij}(v) = \frac{1}{2}(\partial_j v_i + \partial_i v_j) = \{ e_{ij}(v) + \Gamma^p_{ij} v_p \} \in L^2(\Omega)$. 0,25

Since $w \in L^2(\Omega)$ implies $\partial_k w \in H^{-1}(\Omega)$, we obtain 0,25

$$\begin{cases} \partial_k v_i \in H^{-1}(\Omega), \\ \partial_j(\partial_k v_i) = \{ \partial_j s_{ik}(v) + \partial_k s_{ij}(v) - \partial_i s_{jk}(v) \} \in H^{-1}(\Omega). \end{cases}$$ 0,5

According to Lemma of J.L. Lions, we deduce that $\partial_k v_i \in L^2(\Omega)$.
Hence $v \in H^1(\Omega)$. 0,5

3. The identity mapping i from $H^1(\Omega)$ equipped with $\|.\|_{1,\Omega}$ into $W(\Omega)$ equipped with $\|.\|$ is injective, continuous:
there exists a constant $c(\Omega)$ such that $\|v\| \leqslant c(\Omega)\|v\|_{1,\Omega}$ $\forall v \in H^1(\Omega)$, 1
and surjective by (ii).

Then the closed graph theorem shows that the inverse i^{-1} is also continuous: there exists a constant $C(\Omega)$ such that 1

$$\|v\|_{1,\Omega} \leqslant C(\Omega)\|v\| \quad \text{for all } v \in H^1(\Omega).$$