Kasdi Merbah Ouargla University 2024-2025 academic year
Mathematics Department

Exercise set 3 - Topology

Exercise 1: Are the following functions metrics on E?

$$1) \quad d(x,y) = \left|x^2 - y^2\right|, \quad E = \mathbb{R}.$$

$$2) \quad d(x,y) = \left|x^3 - y^3\right|, \quad E = \mathbb{R}.$$

$$3) \quad d(x,y) = e^{x-y}, \quad E = \mathbb{R}.$$

$$4) \quad d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|, \quad E = \mathbb{R}^*.$$

$$5) \quad d(x,y) = |x - 3y|, \quad E = \mathbb{R}.$$

Exercise 2: Let $E = \{x, y, z\}$, and $d : E \times E \longrightarrow \mathbb{R}$ a map defined by

$$\begin{cases} d(x,x) = d(y,y) = d(z,z) = 0, \\ d(x,y) = d(y,x) = 1, \\ d(y,z) = d(z,y) = 2, \\ d(x,z) = d(z,x) = 4. \end{cases}$$

Is d a metric on E?

Exercise 3: Let d be a metric on E.
1- Prove that the following functions are also metrics on E:

$$\begin{aligned} (a) \qquad d_1(x,y) &= \min\{d(x,y), 1\}, \\ (b) \qquad d_2(x,y) &= \frac{d(x,y)}{1 + d(x,y)}. \end{aligned}$$

2- Show that a set $O \subset E$ is open with repect to d if and only if it is open with respect to d_1 if and only if it is open with respect to d_2.
3- If (E, d) is complete, show that (E, d_2) is complete.

Exercise 4: Let E be a non-empty set. Define on $E \times E$ the map

$$d(x,y) = \begin{cases} 0, & \text{si } x = y \\ 1, & \text{si } x \neq y \end{cases}.$$

1- Show that d is a metric (discrete metric) on E.
2- Let $a \in E$ and $r \in \mathbb{R}_+^*$. Determine the open ball $B_d(a, r)$, the closed ball $\overline{B}_d(a, r)$ and the sphere $S_d(a, r)$.

Exercise 5: Let (E,d) be a metric space, a point of E and A a non-empty set of E. Show that:

$$a \in \overline{A} \iff d(a,A) = 0.$$

Exercise 6: Let (E_1,d_1) and (E_2,d_2) be two metric spaces, and f a map from E_1 to E_2.
1- Show that:

$$\begin{array}{lcl} f \text{ uniformly continuous} & \iff & \\ \forall (x_n)_{n\in\mathbb{N}},\ (y_n)_{n\in\mathbb{N}} \text{ sequences of } E_1 & : & \lim\limits_{n\to+\infty} d_1(x_n,y_n) = 0 \implies \lim\limits_{n\to+\infty} d_2(f(x_n),f(y_n)) = 0. \end{array}$$

2- Deduce that the function $f(x) = \sin\dfrac{1}{x}$ is not uniformly continuous on $]0,+\infty[$.
(Take: $x_n = \dfrac{1}{2n\pi}$, $y_n = \dfrac{1}{2n\pi + \frac{\pi}{2}}$, $n \in \mathbb{N}^*$).

Exercise 7: Let (E_1,d_1) and (E_2,d_2) be two metric spaces, and let f be a uniformly continuous mapping from E_1 to E_2.
Show that the image of any Cauchy sequence in E_1 under f is a Cauchy sequence in E_2.

Exercise 8 : Let (E,d) be a metric space, A and B two non-empty disjoint closed sets of E. Show that there exists $f : E \longrightarrow \mathbb{R}$ continuous such that:

$$f|_A = 0; \qquad f|_B = 1; \quad 0 \le f \le 1.$$