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Exercise set 2 - Topology

Exercise 1: Let (E,T) be a topological space and let $A \subset E$. Show that

$$\overline{A^c} = (int(A))^c \quad \text{and} \quad \left(\overline{A}\right)^c = int\,(A^c)\,.$$

Exercise 2: Let $\mathbb{R}$ be endowed with its standard topology. Let A be a topological subspace of $\mathbb{R}$.
(a) Is $\{3\}$ open in $A = [0,1) \cup \{3\}$?
(b) Are $[0,1)$ and $(0,1)$ open in $A = [0,1]$?
(c) Let $n \in \mathbb{N}$. Is $\{n\}$ open in $A = \mathbb{N}$?
(d) Show that $[0,1]$ and $(2,3)$ are both open in $A = [0,1] \cup (2,3)$. What can you deduce from that?
(e) What is the closure of $(0, \frac{1}{2})$ in $A = (0,1]$?

Exercise 3: Consider the topological space (E,T) defined by

$$E = \{a,b,c,d\} \text{ and } T = \{\varnothing, E, \{a\}, \{b\}, \{a,b\}, \{b,c,d\}\}\,.$$

Let $f : E \longrightarrow E$ be a mapping defined by

$$f(a) = b,\ f(b) = d,\ f(c) = b,\ f(d) = c.$$

Is f continuous at a? at b? at c? at d?

Exercise 4: Let (E_1,T_1) be a topological space and $A \subset E_1$. Let $E_2 = \{0,1\}$be equipped with the topology $T_2 = \{\emptyset, \{1\}, E_2\}$. Define a function $f : E_1 \longrightarrow E_2$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Show that f is continuous on E_1 if and only if A is open in E_1.

Exercise 5: Let (E_1,T_1) and (E_2,T_2) be two topological spaces. Let $f : E_1 \longrightarrow E_2$ be a mapping. Show the following:
(a) f is continuous if and only if $f^{-1}(A^o) \subset \left[f^{-1}(A)\right]^o$ for all $A \subset E_2$.
(b) f is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset E_1$.
(c) f is closed if and only if $\overline{f(A)} \subset f(\overline{A})$ for all $A \subset E_1$.
(d) f is open if and only if $f(A^o) \subset [f(A)]^o$ for all $A \subset E_1$.

Exercise 6: Let (E_1,T_1) and (E_2,T_2) be two topological spaces. Let $f : E_1 \longrightarrow E_2$ be a continuous and one-to-one mapping.
(a) Show that if E_2 is **Hausdorff**, then so is E_1.
(b) Give a counterexample showing that the hypothesis of the continuity cannot merely be dropped.
(c) Give a counterexample showing that the hypothesis of the injectivity cannot be completely eliminated.

Exercise 7: Let E be a Hausdorff topological space.
(a) Show that every finite subset is closed.
(b) Show that every topological subspace of E is Hausdorff.
(c) Show that every topological space homeomorphic to E is Hausdorff.
(d) Let F be a topological space. Show that if there exists a continuous injective map from F to E, then F is Hausdorff.

Exercise 8: Let (E,T) be a topological space. The **diagonal** of E is defined to be the set

$$\Delta = \{(x,x) : x \in E\}.$$

Show that E is **Hausdorff** if and only if its diagonal is closed.