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Exercise set 1 - Topology

Exercise 1: Let $E = \{a, b, c, d\}$ a set with four points. Which of the following ones are topologies for E?
(a) $\{\emptyset, E, \{a\}, \{b\}, \{a, c\}, \{a, b\}, \{a, b, c\}\}$
(b) $\{\emptyset, E, \{a\}, \{b\}, \{a, b\}, \{b, d\}\}$
(c) $\{\emptyset, E, \{a, c, d\}, \{b, c, d\}\}$

Exercise 2: For each $x \in \mathbb{R}$, let $I_x = (x, \infty)$, and let $I_\infty = \emptyset$ and $I_{-\infty} = \mathbb{R}$. Check that

$$T = \{I_x : x \in \mathbb{R} \cup \{-\infty, \infty\}\}$$

defines a topology on $\mathbb{R}$.

Exercise 3: Let E be a set and let p be an element of E. Check that

$$T = \{A \subseteq E : p \notin A \ or \ E \backslash A \ is \ finite\}$$

defines a topology on E.

Exercise 4: Let T be a topology on the set $E = \{a, b, c\}$. Show that if the singletons $\{a\}$, $\{b\}$ and $\{c\}$ are open in T, then T is the discrete topology.

Exercise 5: Prove that any intersection $\cap T_i$ of topologies T_i on the same set E is a topology. Show that it is not true, in general, for unions.

Exercise 6: The goal of this exercise is to give some equivalent characterizations for the **interior** of a set. Let E be a topological space and let F be a subset of E. Moreover, let:
(i) $int(F) = \{x \in E$ there exists O open such that $x \in O \subseteq F$ $\}$;
(ii) F_1 be the maximal open set that is contained in F (if it exists);
(iii) F_2 be the union of all the open sets that are contained in F.
Show that F_1 exists and $int(F) = F_1 = F_2$.

Exercise 7: The goal of this exercise is to give some equivalent characterizations for the **closure** of a set. Let E be a topological space and let F be a subset of E. Let:
(i) $\overline{F} = int(F) \cup \{x \in E|$ for each open O such that $x \in O$, $O \cap F \neq \emptyset \neq O \cap (E \backslash F)\}$;
(ii) F_1 be the minimal closed set that contains F (if it exists);
(iii) F_2 be the intersection of all the closed sets that contain F;
(iv) $F_3 = E \backslash int(E \backslash F)$.
Show that F_1 exists and $\overline{F} = F_1 = F_2 = F_3$.

Exercise 8: Give an example of two subsets A and B of $\mathbb{R}$ such that

$$A \cap B = \emptyset, \qquad \overline{A} \cap B \neq \emptyset, \qquad A \cap \overline{B} \neq \emptyset.$$

Exercise 9: Let A and B be subsets of a topological space E. Show that:
(i) $int(A) \cap int(B) = int(A \cap B)$;
(ii) $int(A) \cup int(B) \subseteq int(A \cup B)$;
(iii) $\overline{A} \cup \overline{B} = \overline{A \cup B}$;
(iv) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$;
(v) Give one example where the equality in part (ii) is satisfied, one where it fails, one where the equality in part (iv) is satisfied and one where it fails.