Module:Optimization 1

–Tutorial 05 –

Exercise 01: Hilbert Spaces

For any integer $N \in \mathbb{N}$, we denote by M_N the subspace of $\ell^2(\mathbb{N}, \mathbb{C})$ formed by sequences $(x_n)_{n\in\mathbb{N}}$ such that: $\sum_{n=0}^{N} x_n = 0$.

1°) Show that the mapping $(x_n)_{n\in\mathbb{N}} \mapsto \sum_{n=0}^{N} x_n$ is linear and continuous from $\mathbb{N}$ to $\mathbb{C}$. What can you deduce about $\mathbb{C}$? Conclude that $\ell^2(\mathbb{N}, \mathbb{C}) = M_N \oplus M_N^{\perp}$.

2°) Let $E = \{(y_n)_{n\in\mathbb{N}} : y_i = y_j \text{ for } 0 \le i < j \le N, \text{ and } y_n = 0 \text{ for } n > N\}$.

a) Show that the orthogonal $M_N^{\perp}$ contains E.

b) Prove that $M_N^{\perp} = E$ (observe that, for $0 \le i \le j \le N$, the sequence (x_n) defined by $x_i = 1$, $x_j = -1$, and $x_n = 0$ otherwise, belongs to M_N).

Exercise 02

Consider the pre-Hilbert space $E = \mathcal{C}([-1, 1], \mathbb{R})$ equipped with the inner product:

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, dx, \quad \forall f, g \in E.$$

Let the sequence of functions $(f_n)_{n\in\mathbb{N}^*}$ be defined by:

$$f_n(x) = \begin{cases} 0, & \text{if } -1 \le x \le \frac{-1}{n}, \\ nx + 1, & \text{if } \frac{-1}{n} \le x \le 0, \\ 1, & \text{if } 0 < x \le 1. \end{cases}$$

1°) Show that $(f_n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_2$ induced by the inner product defined above.

2°) Prove that $(E, \|\cdot\|_2)$ is not complete.

3°) Define $F = \{f \in E : f(x) = 0\, \forall x \in [-1, 0]\}$ and $G = \{f \in E : f(x) = 0\, \forall x \in [0, 1]\}$. Show that $F \cap G = \{0\}$.

4°) Are the closed subspaces F and G complementary? Conclude.

Exercise 03

Let H be a Hilbert space, and let $x \in H$, $(e_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in H, and let F be the vector subspace spanned by $(e_n)_{n\in\mathbb{N}}$. Let F_n be the vector subspace spanned by $\{e_1, \ldots, e_n\}$, and let P_{F_n} be the orthogonal projection onto F_n.

1°) Show that:

$$P_{F_n}(x) = \sum_{k=1}^{n} \langle x, e_k\rangle e_k.$$

2°) Prove that:

$$\|x - P_{F_n}(x)\|^2 = \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k\rangle|^2.$$

3°) Deduce that:

$$\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2.$$

4°) We define:

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 + \|x - P_{F_n}\|^2 = \|x\|^2.$$