Module:Normed vector Space

–Tutorial 04 –

Exercise 01:

Let N_1 and N_2 be two norms on the vector space E. Show that N_1 and N_2 are equivalent if and only if the identity

$$Id : (E, N_1) \to (E, N_2)$$

and

$$Id : (E, N_2) \to (E, N_1)$$

are continuous.

Exercise 02:

Determine whether the linear mapping $T : (E, N_1) \to (F, N_2)$ is continuous in the following cases:

1. $E = C([0,1], \mathbb{R})$ with $\|f\|_1 = \int_0^1 |f(t)|\,dt$, and $T : (E, \|.\|_1) \to (E, \|.\|_1)$, $f \mapsto fg$, where $g \in E$ is fixed.
2. $E = \mathbb{R}[X]$ with $\|\sum_{k\geq 0} a_k X^k\| = \sum_{k\geq 0} |a_k|$, and $T : (E, \|\cdot\|) \to (E, \|\cdot\|)$, $P \mapsto P'$.
3. $E = \mathbb{R}^n[X]$ with $\|\sum_{n_k=0} a_k X^k\| = \sum_{n_k=0} |a_k|$, and $T : (E, \|\cdot\|) \to (E, \|\cdot\|)$, $P \mapsto P'$.
4. $E = \mathbb{R}[X]$ with $\|\sum_{k\geq 0} a_k X^k\| = \sum_{k\geq 0} k!|a_k|$, and $T : (E, \|\cdot\|) \to (E, \|\cdot\|)$, $P \mapsto P'$.
5. $E = C([0,1], \mathbb{R})$ with $\|f\|_2 = \left(\int_0^1 |f(t)|^2\,dt\right)^{1/2}$, $F = C([0,1], \mathbb{R})$ with $\|f\|_1 = \int_0^1 |f(t)|\,dt$, and $T : (E, \|.\|_2) \to (F, \|.\|_1)$, $f \mapsto fg$, where $g \in E$ is fixed.

Exercise 03:

Let $E = C([0,1], \mathbb{R})$ be equipped with the norm $\|f\|_1 = \int_0^1 |f(t)|\,dt$, where $f \in E$. Consider the endomorphism ϕ of E defined by $\phi(f)(x) = \int_0^x f(t)\,dt$.

(a) Justify the terminology: "ϕ is an endomorphism of E."
(b) Prove that ϕ is continuous.
(c) For $n \geq 0$, consider f_n an element of E defined by $f_n(x) = ne^{-nx}$ for $x \in [0,1]$. Calculate $\|f_n\|_1$ and $\|\phi(f_n)\|_1$.
(d) Determine $\|\phi\|_{\text{op}}$. Is ϕ injective? surjective?
(e) What are the eigenvalues of ϕ?

Exercise 04:

Let $E = \mathbb{R}[X]$ be equipped with the norm $\|\sum_i a_i X^i\| = \sum_i |a_i|$. Is the linear mapping $\phi : (E, \|\cdot\|) \to (E, \|\cdot\|)$, $P(X) \mapsto P(X+1)$, continuous on E? Is the linear mapping $\psi : (E, \|\cdot\|) \to (E, \|\cdot\|)$, $P \mapsto AP$, where A is a fixed element of E, continuous on E?

Exercise 05:

Let $E = M_n(\mathbb{R})$ be equipped with the norm N defined for any $A = (a_{i,j})_{1\leq i,j\leq n}$ by $N(A) = \sup_{1\leq i\leq n}\left\{\sum_{j=1}^n |a_{i,j}|\right\}$ (it is assumed to be a norm). Prove that the trace function $\text{Tr} : E \to \mathbb{R}$ is continuous, and calculate its norm.

Exercise 06:

Let $E = C([0,1])$ be equipped with $\|\cdot\|_\infty$, and let $F = C^1([0,1])$ be equipped with $\|f\|_F = \|f\|_\infty + \|f'\|_\infty$. Consider the mapping $T : E \to F$ defined by $Tf(x) = \int_0^x f(t)\,dt$. Prove that T is continuous and calculate its norm.