Subject: Normed Vector Space
–Tutorial 01 –

Exercise 01
We define three functions on $\mathbb{R}^2$ as follows:

$$N_1((x,y)) = |x| + |y|,\ N_2((x,y)) = \sqrt{x^2+y^2},\ N_\infty((x,y)) = \max(|x|,|y|).$$

1°) Prove that N_1, N_2, N_∞ define 3 norms on $\mathbb{R}^2$.
2°) Prove that we have: $\forall \alpha \in \mathbb{R}^2,\ N_\infty(\alpha) \leq N_2(\alpha) \leq N_1(\alpha) \leq 2N_\infty(\alpha). Are$N$_1$, N_2, and N_∞ equivalent?
3°) Draw the closed unit balls associated with these norms.

Exercise 02
Let E be the vector space of continuous functions on $[0,1]$ with values in $\mathbb{R}$. We define for $f \in E$

$$\|f\|_\infty = \sup\{|f(x)|;\ x \in [0,1]\},\ \|f\|_1 = \int_0^1 |f(t)|dt.$$

1°) Verify that $\|.\|_\infty$ and $\|.\|_1$ are two norms on E.
2°) Show that, for all $f \in E$, $\|f\|_1 \leq \|f\|_\infty$. Using the sequence of functions $f_n(x) = x^n$,
3°) prove that these two norms are not equivalent.

Exercise 03
Let $(E, \|\cdot\|)$ be a normed vector space.
1°) a) Prove that, for all $x, y \in E$, we have

$$\|x\| + \|y\| \leq \|x+y\| + \|x-y\|.$$

b) Deduce that

$$\|x\| + \|y\| \leq 2\max(\|x+y\|, \|x-y\|).$$

Can the constant 2 be improved? Now, we assume that the norm is induced by an inner product.
2°) a) Prove that, for all $x, y \in E$, we have

$$(\|x\| + \|y\|)^2 \leq \|x+y\|^2 + \|x-y\|^2.$$

b) Deduce that

$$\|x\| + \|y\| \leq \sqrt{2}\max(\|x+y\|, \|x-y\|).$$

Can the constant $\sqrt{2}$ be improved?

Exercise 04
Show that if two balls $\bar{B}(a,r)$ and $\bar{B}(a',r')$ in a normed vector space $(E, \|\cdot\|)$, $E \neq \{0\}$, are equal, then $a = a'$ and $r = r'$.

Exercise 05
Let E be a normed vector space. For $a \in E$ and $r > 0$, we denote $\bar{B}(a,r)$ the closed ball centered at a with radius r. We fix $a \in E$ and $r, s > 0$.
1°) Suppose that $\bar{B}(a,r) \subset \bar{B}(b,s)$. Prove that $\|a-b\| \leq s - r$.
2°) Suppose that $\bar{B}(a,r) \cap \bar{B}(b,s) = \varnothing$. Prove that $\|a-b\| > r + s$.