Subject: Normed Vector Space Series

–Tutorial 02–

Exercise 01

Let $a_1, \ldots, a_n$ be real numbers and $N : \mathbb{R}^n \to \mathbb{R}$ defined by

$$N(x_1, \ldots, x_n) = a_1|x_1| + \cdots + a_n|x_n|.$$

Provide a necessary and sufficient condition on the a_k for N to be a norm on $\mathbb{R}^n$.

Exercise 02

Let N_1 and N_2 be two norms on a vector space E. Define $N = \max(N_1, N_2)$. Show that N is a norm on E.

Exercise 03

Let $a, b > 0$. Define, for all $(x, y) \in \mathbb{R}^2$, $N(x, y) = \sqrt{a^2x^2 + b^2y^2}$.
1°) Prove that N is a norm.
2°) Draw the ball centered at 0 with radius 1.
3°) Determine the smallest number $p > 0$ such that $N \leq p\|.\|_2$ and the largest number q such that $q\|.\|_2 \leq N$.

Exercise 04

For any $x = (a, b) \in \mathbb{R}^2$, define

$$N(x) = \sqrt{a^2 + 2ab + 5b^2}.$$

Show that N is a norm on $\mathbb{R}^2$.

Exercise 05

Let $x, y, p, q \in \mathbb{R}_+^*$ such that $1/p + 1/q = 1$, and $a_1, \ldots, a_n, b_1, \ldots, b_n$ be $2n$ strictly positive real numbers.
1°) Show that

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q.$$

2°) Suppose in this question that $\sum_{i=1}^n a_i^p = \sum_{i=1}^n b_i^q = 1$. Show that $\sum_{i=1}^n a_ib_i \leq 1$.
3°) Deduce the magnificent Hölder's inequality:

$$\sum_{i=1}^n a_ib_i \leq \left(\sum_{i=1}^n a_i^p\right)^{1/p} \left(\sum_{i=1}^n b_i^q\right)^{1/q}.$$

4°) Further assume that $p > 1$. Derive Minkowski's inequality from Hölder's inequality:

$$\left(\sum_{i=1}^n (a_i + b_i)^p\right)^{1/p} \leq \left(\sum_{i=1}^n a_i^p\right)^{1/p} + \left(\sum_{i=1}^n b_i^p\right)^{1/p}.$$

5°) Define for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}.$$

Show that $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$.

Exercise 06

Let N be the map from $\mathbb{R}^2$ to $\mathbb{R}$:

$$(x, y) \mapsto \sup_{t\in\mathbb{R}} \frac{|x + ty|}{\sqrt{1 + t^2}}$$

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1°) Show that N is a norm on $\mathbb{R}^2$.
2°) Compare it with the Euclidean norm.
3°) Explain.