Module:Normed vector Spaces

–Tutoria 03 –

Banach Space

Exercise 01:

Let X be a set. We denote $B(X,\mathbb{R})$ as the vector space of bounded functions from X to $\mathbb{R}$. We equip $B(X,\mathbb{R})$ with a norm $\|\cdot\|$ defined as follows:

$$\forall f \in B(X,\mathbb{R}), \quad \|f\| = \sup_{x\in X} |f(x)|.$$

- Show that $(B(X,\mathbb{R}), \|\cdot\|)$ is a Banach space.

Exercise 02:

Let $E = C^1([0,1])$ be equipped with the norm

$$N(f) = \|f\|_\infty + \|f'\|_\infty.$$

- Show that (E, N) is a complete space.

Exercise 03:

Let E be the vector space of continuous functions from $[-1,1]$ to $\mathbb{R}$. We define a norm on E by setting $\|f\|_1 = \int_{-1}^{1} |f(t)|\, dt$. We will show that E equipped with this norm is not complete. For this purpose, we define a sequence $(f_n)_{n\in\mathbb{N}^*}$ as follows:

$$f_n(t) = \begin{cases} -1 & \text{if } -1 \le t \le -\frac{1}{n}, \\ nt & \text{if } -\frac{1}{n} \le t \le \frac{1}{n}, \\ 1 & \text{if } \frac{1}{n} \le t \le 1. \end{cases}$$

- Verify that $f_n \in E$ for all $n \ge 1$.
- Show that

  $$\|f_n - f_p\|_1 \le \sup(\frac{2}{n}, \frac{2}{p})$$

  and deduce that (f_n) is Cauchy.
- Assume that there exists a function $f \in E$ such that (f_n) converges to f in $(E, \|\cdot\|_1)$. Show that in this case, we have

  $$\lim_{n\to+\infty} \int_{-\alpha}^{-1} |f_n(t) - f(t)|\, dt = 0$$

  and

  $$\lim_{n\to+\infty} \int_{1}^{\alpha} |f_n(t) - f(t)|\, dt = 0$$

  for all $0 < \alpha < 1$.

  Show that we also have

  $$\lim_{n\to+\infty} \int_{-\alpha}^{-1} |f_n(t) + 1|\, dt = 0$$

  and

  $$\lim_{n\to+\infty} \int_{1}^{\alpha} |f_n(t) - 1|\, dt = 0$$

  for all $0 < \alpha < 1$.

  Conclude that

  $$f(t) = \begin{cases} -1 & \forall t \in [-1, 0[, \\ 1 & \forall t \in]0, 1]. \end{cases}$$

  Conclusion.