Problems set — Variational Inequalities

Exercise 1. Projection Theorem.
Let K be a non-empty closed convex set in a Hilbert space V. We say that y is the projection of x on K if

$$\|x-y\| = \inf_{z\in K} \|x-z\|$$

We denote $y = P_K x$.

1. Show that if K is a closed convex set in V, then for all $x \in V$, there exists a unique projection $y \in K$.
2. Show that
$$y = P_K x \Leftrightarrow (P_K x - x, P_K x - z) \leq 0, \forall z \in K.$$
3. Show that the mapping P_K is a contraction, i.e.,
$$\|P_K x_1 - P_K x_2\| \leq C\|x_1 - x_2\|, \forall x_1, x_2 \in V, 0 \leq C \leq 1.$$
4. Show that the mapping P_K is monotone, i.e.,
$$(P_K x_1 - P_K x_2, x_1 - x_2) \geq 0, \forall x_1, x_2 \in V.$$
5. If K is a closed subspace of V, then P_K is linear and $V = K \oplus K^{\perp}$.

Exercise 2. Stampacchia's Theorem.
Let $a(.,.)$ be a continuous bilinear form and coercive on a Hilbert space V. Let K be a nonempty closed convex set in V, and let $f \in V'$. Consider the following problem:

$$(\mathcal{P}) \begin{cases} \text{Find } u \in K \text{ such that} \\ a(u, v-u) \geq (f, v-u), \forall v \in K. \end{cases}$$

1. Show that problem $\mathcal{P}$ has a unique solution.
2. Show that the mapping: $f \mapsto u$ is Lipschitz and satisfies
$$\|u_1 - u_2\| \leq \frac{1}{\alpha}\|f_1 - f_2\|,$$
where $f_1 \mapsto u_1$, $f_2 \mapsto u_2$, and α is the coercivity constant.

Exercise 3. Application of Stampacchia's Theorem.

Let Ω be a bounded open set in $\mathbb{R}^N$ with boundary Γ, $f \in L^2(\Omega)$, n the exterior normal on Γ. Consider the following problem for u:

$$(PC) \begin{cases} -\Delta u + u = f \text{ in } \Omega, \\ u \geq 0, \frac{\partial u}{\partial n} \geq 0, u\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma \end{cases}$$

1. Show that if u is a solution of problem (PC), then it satisfies the following variational inequality (PV):
$$a(u, v-u) \geq L(v-u), \forall v \in K,$$
where
$$a(u,v) = \int_\Omega \nabla u \nabla v + uv dx,\ L(v) = \int_\Omega fv dx,\ K = \{v \in H^1(\Omega), v \geq 0 \text{ on } \Gamma\}.$$
2. Suppose that if u is a solution of (PV) and is regular enough, then u satisfies problem (PC).
3. Show that problem (PC) has a unique weak solution (in the sense of problem (PV)).

Exercise 4. Penalization Method.
Let $a(.,.)$ be a continuous bilinear form and coercive on a Hilbert space V. Let K be a nonempty closed convex set in V, $0 \in K$, $f \in V'$. Consider the following variational inequality:
$$a(u, v-u) \geq L(v-u), \forall v \in K. \tag{1}$$
Let $B = I - P_K$, where P_K is the projection in V onto K.

1. Verify that:
   - $(Bv, v) \geq 0, \forall v \in V$.
   - $B = 0$ on K.
   - $(Bu - Bv, u - v) \geq 0$ (monotonicity).
   - $(Bv, v) = 0 \Leftrightarrow v \in K$.
2. Show the existence and uniqueness of $u_\varepsilon \in V$ as a solution to the equation
$$a(u_\varepsilon, v) + \frac{1}{\varepsilon}(Bu_\varepsilon, v) = L(v), \forall v \in V.$$
3. Show that:
   - $\|u_\varepsilon\| \leq C$, where C is independent of ε.
   - $(Bu_\varepsilon, u_\varepsilon) \leq \varepsilon C$
   - $(Bu_\varepsilon, v) \leq \varepsilon C \|v\|, \forall v \in V$.
4. Show that the weak limit of the sequence u_ε is a solution of the variational inequality (2).

Exercise 5. Approximation interne
Let $a(.,.)$ be a continuous bilinear form and coercive on a Hilbert space V. Let K be a nonempty closed convex set in V, and $f \in V'$. Consider the following variational inequality (P):
$$(P)\begin{cases} u \in K \\ a(u, v-u) \geq (f, v-u), \forall v \in K. \end{cases} \tag{2}$$
Suppose that there exists a sequence of finite-dimensional spaces $V_h \subset V$, such that $\dim V_h < \infty$, and $\forall v \in U$, there exists $(v_h) \subset V_h$ such that $v_h \to v$ in V as $h \to 0$.

Suppose also that there exists a sequence of convex closed sets K_h in V_h, such that
$$\begin{cases} \text{For all } v \in K, \text{ there exists } (v_h) \subset K_h \text{ such that } v_h \to v \text{ in } V, \\ \text{If } u_h \in K_h \text{ and } u_h \rightharpoonup u \text{ in } V \text{ as } h \to 0, \text{ then } u \in K. \end{cases} \tag{3}$$

Define the following problem:

$$(P_h)\begin{cases} u_h \in K_h \\ a(u_h, v_h - u_h) \geq (f, v_h - u_h), \forall v_h \in K_h. \end{cases} \quad (4)$$

Show that

$$\lim_{h\to 0} \|u_h - u\|_V = 0,$$

where u_h is the solution to (P_h) and u is the solution to (P).

Exercise 6. **(Proximity Operator)** Let V be a Hilbert space, $\varphi : V \to \bar{\mathbb{R}}$ be a proper, convex, and lower semi-continuous function. The proximity operator associated with the function φ is denoted by $Prox_\varphi$ and satisfies

$$\begin{array}{cccc} Prox_\varphi & : & V & \to & V \\ & w & \mapsto & u = Prox_\varphi(w) \end{array}$$

where u is the unique minimizer of the function

$$\Phi_w(v) = \frac{1}{2}\|v\|^2 + \varphi(v) - (w, v), \forall v \in V \quad (5)$$

1. Show that:

$$u = Prox_\varphi(w) \Leftrightarrow (u, v - u) + \varphi(v) - \varphi(u) \geq (w, v - u), \forall v \in V. \quad (6)$$

2. Show that the function $Prox_\varphi$ is monotone, i.e.,

$$(Prox_\varphi(w_1) - Prox_\varphi(w_2), w_1 - w_2) \geq 0, \forall w_1, w_2 \in V. \quad (7)$$

3. Show that the function $Prox_\varphi$ is contractive, i.e.,

$$\|Prox_\varphi(w_1) - Prox_\varphi(w_2)\| \geq \|w_1 - w_2\|, \forall w_1, w_2 \in V. \quad (8)$$

4. Conclude that

$$u = Prox_\varphi(u) \Leftrightarrow \varphi(u) \leq \varphi(v), \forall v \in V. \quad (9)$$

Exercise 7. *Let V and φ be defined as in Exercise 1. Let K be a non-empty closed convex subset of V, $a(.,.) : V \times V \to \mathbb{R}$ be a continuous bilinear form, and $f \in V$. We consider the problem:*

$$(P)\begin{cases} \text{Find } u \in K \text{ such that} & (10) \\ a(u, v - u) + \varphi(v) - \varphi(u) \geq (f, v - u), \forall v \in K. & (11) \end{cases}$$

1. *Suppose that a is positive, i.e., $a(v, v) \geq 0, \forall v \in V$. Show that problem (P) is equivalent to the problem:*

$$(Q)\begin{cases} \text{Find } u \in K \text{ such that} & (12) \\ a(v, v - u) + \varphi(v) - \varphi(u) \geq (f, v - u), \forall v \in K. & (13) \end{cases}$$

2. *Show that the set of solutions of (P) is a closed convex subset of K.*

3. *Suppose further that the form a is symmetric. Show that (P) is equivalent to the following minimization problem:*

$$(M)\begin{cases} \text{Find } u \in K \text{ such that} & (14) \\ J(u) \leq J(v), \forall v \in K. & (15) \end{cases}$$

where the function $J : V \to \mathbb{R}$ is defined as:

$$J(v) = \frac{1}{2}a(v, v) + j(v) - (f, v), \forall v \in V. \quad (16)$$

Exercise 8. *Consider a continuous and positive bilinear form $a(.,.)$ on a Hilbert space $V \times V$. Let K be a non-empty closed convex subset of V, and $f \in V'$. We introduce the following problem:*

$$(\mathcal{P}) \begin{cases} \text{Find } u \in K, \text{ such that} \\ a(u, v-u) \geq (f, v-u), \text{ for all } v \in K. \end{cases}$$

1. *Show that problem (P) is equivalent to the problem*

$$(Q) \begin{cases} \text{Find } u \in K, \text{ such that} & (17) \\ a(v, v-u) \geq (f, v-u), \text{ for all } v \in K. & (18) \end{cases}$$

2. *Prove that the set of solutions $\mathcal{S} = \{u \in K / a(u, v-u) \geq (f, v-u), \text{ for all } v \in K\}$ of problem $(\mathcal{P})$ is a closed convex set.*

3. *Let $\beta(.,.)$ be a continuous and coercive bilinear form on $V \times V$, and $g \in V'$. For $\varepsilon > 0$, we define the problem*

$$(\mathcal{P}) \begin{cases} \text{Find } u^\varepsilon \in K, \text{ such that} \\ a(u^\varepsilon, v-u^\varepsilon) + \varepsilon\beta(u^\varepsilon, v-u^\varepsilon) \geq (f+\varepsilon g, v-u^\varepsilon), \text{ for all } v \in K. \end{cases}$$

   Show that problem $(\mathcal{P}^\varepsilon)$ has a unique solution u^ε.

4. *Prove that*

$$\mathcal{S} \neq \emptyset \iff \exists C > 0, \|u^\varepsilon\| \leq C, \tag{19}$$

   C is independent of ε.

5. *Assuming that $\|u^\varepsilon\| \leq C$, demonstrate that $u^\varepsilon \to u^0$ as $\varepsilon \to 0$, where u^0 is the unique solution of the problem*

$$(\mathcal{P}) \begin{cases} \text{Find } u^0 \in \mathcal{S}, \text{ such that} \\ \beta(u^0, v-u^0) \geq (g, v-u^0), \text{ for all } v \in \mathcal{S}. \end{cases}$$

Exercise 9. *Let V be a Hilbert space, and $\varphi(.,.) : V \times V \to \bar{\mathbb{R}}$ be a convex function that is lower semi-continuous with respect to the first component, i.e., if $v_n \to v$, then $\liminf_{n\geq 1} \varphi(v_n, w) \geq \varphi(v, w)$ for all $w \in V$. Additionally, $\varphi(tu + (1-t)v, w) \leq t\varphi(u, w) + (1-t)\varphi(v, w)$ for all $u, v, w \in$, and $t \in [0,1]$. Let K be a non-empty closed convex subset of V, $a(.,.) : V \times V \to \mathbb{R}$ be a continuous bilinear form, and $f \in V$.*

We consider the problem:

$$(P) \begin{cases} \text{Find } u \in K, \text{ such that} & (20) \\ a(u, v-u) + \varphi(v, u) - \varphi(u, u) \geq (f, v-u), \text{ for all } v \in K. & (21) \end{cases}$$

1. *Suppose that a is positive, i.e., $a(v, v) \geq 0$, for all $v \in V$. Show that problem (P) is equivalent to the problem*

$$(Q) \begin{cases} \text{Find } u \in K, \text{ such that} & (22) \\ a(v, v-u) + \varphi(v, u) - \varphi(u, u) \geq (f, v-u), \text{ for all } v \in K. & (23) \end{cases}$$

2. *Prove that the set of solutions of (P) is a closed convex subset of K.*

3. *If a is also symmetric, show that (P) is equivalent to the following minimization problem:*

$$(M) \begin{cases} \text{Find } u \in K, \text{ such that} & (24) \\ J(u) \leq J(v), \text{ for all } v \in K. & (25) \end{cases}$$

   where the function $J : V \to \mathbb{R}$ is defined as:

$$J(v) = \frac{1}{2}a(v, v) + \varphi(v, v) - (f, v), \text{ for all } v \in V. \tag{26}$$

Good luck!